

A NOTE ON RECURRENCE SEQUENCES

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ABSTRACT. In this note we provide a simple formula of general term of recurrent sequence.

Suppose that the sequence $\{u_h\}$ satisfies the relation

$$u_{h+n} = \sum_{i=1}^n s_i u_{n+h-i} \quad \text{for } h = 0, 1, 2, \dots$$

for some complex numbers s_1, \dots, s_n , $s_n \neq 0$ and for $h = 0, 1, \dots$. Taking $h = 0$ we see that u_n is in the ring $\mathbb{Z}[u_0, \dots, u_{n-1}, s_1, \dots, s_n]$, by using the induction argument we show that all the terms in the sequence belong to this ring.

Let

$$X^n - \sum_{i=1}^n s_i X^{n-i} = \prod_{i=1}^m (1 - \alpha_i X)^{n_i}$$

with distinct nonzero roots α_i . It well known that there exist $A_i(X)$ which are polynomials of degree $n_i - 1$ for positive integers n_i , $i \in [1, m]$ such that

$$u_h = \sum_{i=1}^m A_i(h) \alpha_i^h$$

for $h = 0, 1, \dots$, see for example [?], for an original work on recurrent sequences see [?]. More concretely, in this note we give a short proof of this result and the explicit formula $\{u_h\}$.

Consider the vector space E of all sequences $\{v_h\}$ such that

$$v_{h+n} = \sum_{i=1}^n s_i v_{n+h-i} \quad \text{for } h = 0, 1, \dots$$

Remark that $\dim E = n$ and that $e^{j,i} = \{h(h-1)\dots(h+1-j)\alpha_i^{h-j}\}$ for $j = 0, \dots, n_i - 1$, $i = 1, \dots, m$ is a basis of E .

Define the projection $\Psi_k : E \longrightarrow \mathbb{C}^{n+1}$ for $k \geq n$ by

$$\Psi_k(\{v_h\}) := (v_0, \dots, v_{n-1}, v_k)^t.$$

From the fact that $\dim E = n$ we get the vectors $\Psi_k(\{u_h\})$, $\Psi_k(e^{j,i})$, $j = 0, \dots, n_i - 1$, $i = 1, \dots, m$ are linearly dependent, thus

$$\det[\Psi_k(\{u_h\}), \Psi_k(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}] = 0$$

which gives

$$(-1)^n u_h \det[\Psi_n(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}] + \det[f, \Psi_k(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}] = 0$$

with $f = (u_0, \dots, u_{n-1}, 0)^t$, from the fact that $e^{j,i} = \{h(h-1)\dots(h+1-j)\alpha_i^{h-j}\}$ for $j = 0, \dots, n_i - 1$, $i = 1, \dots, m$ is a basis of E we get $\det[\Psi_n(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}] \neq 0$

thus

$$u_h = (-1)^{n+1} \frac{\det[f, \Psi_k(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}]}{\det[\Psi_n(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}]}$$

Also the (ordinary) generating function of the sequence $\{u_h\}$, is given by

$$\sum_{h=0}^{\infty} u_h x^h = (-1)^{n+1} \frac{\det[f, f_{j,i}(x)_{0 \leq j \leq n_i-1, 1 \leq i \leq m}]}{\det[\Psi_n(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}]}$$

with

$$f_{j,i}(x) = (\Psi_n(e^{j,i})^t, \frac{j!x^j}{(1 - \alpha_i x)^{j+1}})^t$$

for $0 \leq j \leq n_i - 1$ and $1 \leq i \leq m$. Of course the vectors are ordered in the same way in the two determinants.

Let I be finite subset of \mathbb{N} and define the projection $\Psi_I : E \longrightarrow \mathbb{C}^I$ by

$$\Psi_I(\{v_h\}) := (v_l)_{l \in I}^t.$$

By using a similar argument as above we get the following theorem

Theorem 0.1. *Let I be a subset of n elements of \mathbb{N} , let $h \notin I$, put $J = I \cup \{h\}$. Then we have*

$$u_h = (-1)^{n+1} \frac{\det[(\Psi_I(\{u_h\}))^t, 0]^t, \Psi_J(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}]}{\det[\Psi_I(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}]}$$

if $\det[\Psi_I(e^{j,i})_{0 \leq j \leq n_i-1, 1 \leq i \leq m}] \neq 0$.

In the case $m = n$ we have $e^{0,i} = \{\alpha_i^h\}$ is a basis of E then we get following corollary

Corollary 0.2. *For all h we have*

$$u_h = (-1)^{n+1} \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i)^{-1} \begin{vmatrix} u_0 & 1 & \dots & 1 \\ u_1 & \alpha_1 & \dots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & \alpha_1^{n-1} & \dots & \alpha_n^{n-1} \\ 0 & \alpha_1^h & \dots & \alpha_n^h \end{vmatrix}$$

Proof. Follows from the theorem by remarking that $I = \{0, 1, \dots, n-1\}$, and $\det[\Psi_I(e^{0,i})_{1 \leq i \leq m}]$ is the Van der Mond determinant with $m = n$, when $h \notin I$. And the equality is also true when $h \in I$, which achieve the proof.

In the next corollary gives u_h as a function of an n fixed terms of the sequence $\{u_h\}$

Corollary 0.3. *Let $k_0 < k_1 < \dots < k_{n-1}$ a sequence of positive integers. Then for all h we have*

$$u_h = (-1)^{n+1} \begin{vmatrix} \alpha_1^{k_0} & \dots & \alpha_n^{k_0} \\ \alpha_1^{k_1} & \dots & \alpha_n^{k_1} \\ \vdots & \ddots & \vdots \\ \alpha_1^{k_{n-1}} & \dots & \alpha_n^{k_{n-1}} \end{vmatrix}^{-1} \begin{vmatrix} u_{k_0} & \alpha_1^{k_0} & \dots & \alpha_n^{k_0} \\ u_{k_1} & \alpha_1^{k_1} & \dots & \alpha_n^{k_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k_{n-1}} & \alpha_1^{k_{n-1}} & \dots & \alpha_n^{k_{n-1}} \\ 0 & \alpha_1^h & \dots & \alpha_n^h \end{vmatrix}$$

when the expression has sense.

Proof. Similar to the proof of Corollary 0.2.

Remark 0.4. `> rsolve(f(0) = 1, f(1) = 1, f(n) = f(n - 1) + f(n - 2), f(n))` gives the generale terms $f(n)$ in Maple.

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